MINIMAL UNCERTAINTY STATES FOR QUANTUM GROUPS

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Abstract

The problem of how to obtain quasi-classical states for quantum groups is examined. A measure of quantum indeterminacy is proposed, which involves expectation values of some natural quantum group operators. It is shown that within any finite dimensional irreducible representation, the highest weight vector and those unitarily related to it are the quasi-classical states.

Quantum groups [1] have been intensively studied in recent years. Their applications have already led to significant progress in statistical mechanics and low dimensional topology. It is also widely believed that quantum groups play some important role in quantum physics as well. In this letter, we will investigate the problem of how to obtain quasi classical states for quantum groups. We will propose a measure of quantum indeterminacy, which involves expectation values of some combinations of Drinfeld's v operator and the universal R - matrix. A quasi-classical state is characterized as having minimal indeterminacy. It will be shown that for any finite dimensional irreducible representation, the highest weight vector and those unitarily related to it are the states having this property. Our study here is an extension to quantum groups of the investigation carried out in [2] some twenty years ago, where the corresponding problem for compact simple Lie groups was resolved by one of us. In the limit $q \to 1$, we recover the results of that publication.

Given a compact simple Lie group G, we denote its Lie algebra by $\mathfrak{g}$. Now there exists a basis $\{e_i\}$, which is self dual with respect to the Killing form, in which the quadratic Casimir operator can be expressed as $C = \sum_i e_i\, e_i$. It was shown in [2] that the following dispersion

$$\sum_i \langle (e_i - \langle e_i \rangle)\,(e_i - \langle e_i \rangle) \rangle \tag{1}$$

was the natural measure of quantum indeterminacy for such an algebra. For any finite dimensional irreducible representation, the highest weight vector and its group orbit, i.e., the Peremolov coherent states [3], proved to be the vectors attaining minimal uncertainty.

Let us rewrite (1) in a more abstract form, so that it will suggests a generalization to quantum groups. Acting on any irreducible representation space with a highest weight λ, C takes the eigenvalue $(\lambda + 2\rho,\ \lambda)$, where 2ρ is the sum of all the positive roots, and $(\, .\, ,\, .\,)$ is the properly normalized inner product of the weight space. However, when acting on the tensor product of two representation spaces, C will no longer be

a scalar multiple of the identity matrix. We express the action of C on the tensor product by $\partial(C)$, and set $Q = [\partial(C) - C \otimes 1 - 1 \otimes C]$. For any unit vector $|\psi\rangle$, the indeterminacy measure (1) can be expressed as

$$\langle\psi|C|\psi\rangle - \langle\psi \otimes \psi|Q|\psi \otimes \psi\rangle, \tag{2}$$

where we have used the self-explanatory notation that $|\psi \otimes \psi\rangle = |\psi\rangle \otimes |\psi\rangle$.

Recall that a quantum group $U_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. It has the structures of a Hopf algebra, namely, it admits a tensor product operation, called the co-multiplication, $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, a co-unit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}$, and an antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, which are all compatible with the algebraic structure of $U_q(\mathfrak{g})$ in an appropriate sense. $U_q(\mathfrak{g})$ also has properties very similar to those of $U(\mathfrak{g})$, e.g., it is generated by raising, lowering and Cartan type of generators. The deformation parameter may be assumed to take various forms, leading to different versions of quantum groups. Here we will take $q = \exp(\eta)$ with η being a real number not equal to 1. For the purpose of this letter, we will allow exponentials of Cartan generators and their polynomials to appear in the algebra as well. Since we will limit ourselves to finite dimensional representations, such things make perfect sense.

We denote by U_q^+ the subalgebra generated by the raising and Cartan elements, and by U_q^- that generated by the lowering and Cartan elements. There exist bases $\{\alpha_t \,|\, t = 1, 2, ...\}$ and $\{\beta_t \,|\, t = 1, 2, ...\}$ for U_q^+ and U_q^- respectively such that

$$R \;=\; \sum_t \alpha_t \otimes \beta_t$$

gives rise to the universal R - matrix of $U_q(\mathfrak{g})$. Furthermore, $U_q(\mathfrak{g})$ admits an involution † satisfying $^\dagger\, S\, ^\dagger\, S = 1$, rendering $\alpha_t^\dagger = \pm\beta_t$. With respect to this involution all finite dimensional representations are unitary.

Let K_ρ be the group like element of $U_q(\mathfrak{g})$ such that $S^2(x) = K_\rho^2\, x\, K_\rho^{-2}$ for all x in $U_q(\mathfrak{g})$. The Drinfeld operator of the quantum group is given by $v = \sum_t S(\beta_t)\, \alpha_t\, K_\rho^{-2}$, which is central, namely, commutes with all the elements of $U_q(\mathfrak{g})$. It is also known that v is fixed by the antipode, i.e., $S(v) = v$, and is invertible with the inverse given by $v^{-1} = \sum_t \beta_t\, K_\rho^2\, \alpha_t$. In a finite dimensional irreducible representation $V(\lambda)$ with highest weight λ, the operator v^{-1} is given by $q^{(\lambda+2\rho,\ \lambda)}\, I$.

In studying the quasi-classical states, we will need the operator v^{-2}, which can be expressed as

$$v^{-2} \;=\; \sum_{r,\,t} \beta_r\, S(\alpha_t)\, S^{-1}(\beta_t)\, \alpha_r.$$

We will also need the operator $(v \otimes v)\Delta(v^{-1}) = R^T R$, where

$$\begin{aligned} R^T R \;&=\; \sum_{r,\,t} \beta_r\, \alpha_t\, \otimes \alpha_r\, \beta_t \\ &=\; \sum_{r,\,t} \beta_r\, S(\alpha_t)\, \otimes \alpha_r\, S(\beta_t). \end{aligned} \tag{3}$$

$R^T R$ acts naturally on the tensor product of two representation spaces. Consider for example the tensor product of $V(\lambda)$ with itself, which can always be decomposed

into a direct sum of finite dimensional irreducible representations

$$V(\lambda)\otimes V(\lambda)=\bigoplus_{i=0}^{L}V(\mu_i),$$

where L is some positive integer which we will not need to know. The $V(\mu_i)$ is an irreducible representation with highest weight μ_i. We will order the μ's in such a way that $\mu_i \geq \mu_j$ if $i<j$. Then clearly, $\mu_0=2\lambda>\mu_i$, for all $i>0$. $R^T R$ in $V(\lambda)\otimes V(\lambda)$, though not being proportional to the identity matrix, can nevertheless be diagonalized, and its eigenvalues are

$$q^{(\mu_i+2\rho,\,\mu_i)-2(\lambda+2\rho,\ \lambda)},\qquad i=0,1,...,L.$$

Let $|\phi\rangle$ be a unit vector of $V(\lambda)$. The following quantity is a natural generalization of (2) to the quantum group setting:

$$\delta_\phi \;=\; \frac{1}{q-q^{-1}}\left[\langle\phi|v^{-2}|\phi\rangle-\langle\phi\otimes\phi|R^T R|\phi\otimes\phi\rangle\right]. \tag{4}$$

Observe that both terms of δ_ϕ are invariant with respect to the quantum group, i.e.,

$$\begin{aligned}\langle\phi|[x,\ v^{-2}]|\phi\rangle &= 0,\\ \langle\phi\otimes\phi|[\Delta(x),\ R^T R]|\phi\otimes\phi\rangle &= 0,\quad \forall x\in U_q(\mathfrak{g}).\end{aligned}$$

Also, we recover from δ_ϕ the dispersion (1) for Lie algebras in the limit $q\rightarrow 1$. We propose δ_ϕ as the measure of quantum indeterminacy, and will call a quantum group state quasi-classical if δ_ϕ is minimized.

In order to minimize δ_ϕ, we observe that the vector $|\phi\otimes\phi\rangle$ can always be written as

$$|\phi\otimes\phi\rangle \;=\; \sum_{i=0}^{L}c_i|\zeta_i\rangle\rangle,$$

where $|\zeta_i\rangle\rangle$ is a unit vector belonging to $V(\mu_i)$, and c_i is a *real number* satisfying the normalization property $\sum_{i=0}^{L}c_i^2=1$. With the help of this expression, we can easily obtain

$$\delta_\phi \;=\; \frac{1}{q-q^{-1}}\left[q^{2(\lambda+2\rho,\ \lambda)}-\sum_{i=0}^{L}c_i^2\,q^{(\mu_i+2\rho,\ \mu_i)-2(\lambda+2\rho,\ \lambda)}\right].$$

Since $\mu_i-\mu_j$, $i<j$, is either zero or a positive integral sum of positive roots of $\mathfrak{g}$, we have

$$(\mu_i+2\rho,\ \mu_i)-(\mu_j+2\rho,\ \mu_j) \;=\; (\mu_i+\mu_j+2\rho,\ \mu_i-\mu_j)\geq 0,\quad i<j.$$

This immediately leads to the conclusion that δ_ϕ reaches its minimum

$$Min(\delta) \;=\; \frac{1}{q-q^{-1}}\left[q^{2(\lambda+2\rho,\ \lambda)}-q^{2(\lambda,\ \lambda)}\right]\geq 0,$$

only when

$$c_0^2 = 1, \quad c_1 = ... = c_L = 0.$$

Therefore, *a state* $|\phi\rangle \in V(\lambda)$ *is quasi-classical if and only if* $|\phi \otimes \phi\rangle$ *belongs to the irreducible component* $V(2\lambda)$ *contained in* $V(\lambda) \otimes V(\lambda)$. It is clearly true that the highest weight vector $|\phi_0\rangle$ (normalized to 1) of $V(\lambda)$ meets this requirement. Given any unit vector $|\phi\rangle$, the irreducibility of $V(\lambda)$ guarantees the existence of a unitary endomorphism U (i.e., $U^\dagger U = 1$) such that $|\phi\rangle = U|\phi_0\rangle$. When $U \otimes U$ commutes with $R^T R$, $|\phi\rangle$ yields a quasi-classical state, and all the quasi-classical states are of this form. In the limit $q \to 1$, these states reduce to Peremolov's coherent states.

A deeper understanding of the quasi-classical states, particularly their underlying geometry, could be gained by studying their properties with respect to the algebra of functions on $U_q(\mathfrak{g})$, but this would take us into the largely unexplored area of noncommutative geometry, which is well beyond the scope of this letter.

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